GEOMETRY AND DYNAMICS ON THE FREE SOLVABLE GROUPS

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Abstract

In this paper we give a geometric realization of free solvable groups, and study its Poisson-Furstenberg boundaries, we also discuss the construction of normal forms in the solvable groups.

Keywords:free solvable groups, homology, fundamental 2-cycle, boundary, normal form.

1 Introduction

Free solvable groups were studied by algebraists in the 40s–50s in works by F.Hall, W.Magnus and others [9, 10, 11]). The main results were concerned with imbedding to the wreath product. Later in 60-s the growth of the lower central series [12], and so called Golod–Shafarevich series were discussed in the literature (e.g. see [13]) were considered. Now these groups have become an object of study from the viewpoint of harmonic analysis and asymptotic characteristics. For this we need in more precise model of these groups. Unfortunately the imbedding to the wreath product which was stuied before is not effective because the image of the groups is difficult to describe explicitly.

The free solvable groups of level two and higher have not exact matrix representations,in a sense they are "infinite-dimensional" (or "big") groups, as we shall see further – in contrary, say, to the free nilpotent groups. In this paper we give a *new topological model of the free metabelian groups*, i.e. the free solvable groups of level two, and calculate their boundaries. Our realization differs from the known ones (Nilsen–Schraer basis of commutant) by its invariance.

Some of wreath products are very similar to the free solvable groups, its had been considered for a long time in the theory of growth and random walks as a natural source of examples and counterexamples. For example in [4] the wreath product $\mathbb{Z} \wr \mathbb{Z}$ was used as an example of a group with superexponential growth of Folner sets. Later in [3] the group $\mathbb{Z}^d \wr \mathbb{Z}$, $d > 2$ — were used as examples of a solvable and thus amenable groups with non-trivial boundary. The term "wreath product" was not used in these works, but the suggested terms "lamplighter group" and "group of dynamical configurations" became popular. Thus the wreath products $\mathbb{Z}^d \wr \mathbb{Z}$ for $d > 2$ give an important example of amenable groups with positive entropy - it was a surprise — a common opinion before was that all amenable groups have zero entropy ([2]). The Furstenberg-Poisson boundary in this example was not calculated completely, only a natural candidate was presented — moreover, it is this candidate that was used to prove positivity of the entropy (owing to the entropy criterion [3]). These kind of examples were used also in the theory of index of von Neumann factors (see [5]). Recently, the wreath products were used by a student A. Dyubina - see [8] in construction of an example of quasi-isometric groups, one of which is solvable, and the other one is not virtually solvable, as well as in construction of an example of intermediate drift growth. But the boundary was not explicitly calculated upto now - we give the precise description as a corollary of the theorem about the boundary for free solvable group.

Now it becomes clear that the *free solvable groups are much more natural and important class of examples than the wreath products with lattices, and the effects which were discovered in the wreath products manifest themselves yet more explicitly in the free solvable groups.*

In this paper we shall give:

a new topological model of metaabelean (= free solvable group of level 2),

a new (geometrical) normal form for the elements of it

and finally we describe the Poisson-Furstenberg boundary of these groups.

Using the same method and reduction to the free solvable group, we also descirbe the boundary of wreath products - old problems which appeared in [3].

2 Topological model of free solvable groups of the level two

We start with a new, as far as we know, model of the free solvable group of level two having a topological interpretation.

Let Sol_d^2 be the free solvable group of level two with d generators, i.e. the universal object of the variety of solvable groups of level two with d generators. This group may be defined in a more constructive way as the factor group of the free group with d generators with respect to the second commutant:

$$Sol_d^2 = F_d/(F_d)''$$

where F_d is free groups with d generators, and G'' is the second comutant of the group G. Sometimes we omit "2" in Sol_d^2 because in this paper we consider solvable group of level two only.

The commutant of this group is an abelian group with infinite number of generators which may be indexed by the elements of the d-dimensional lattice. Namely, they are the images of the commutators $[x_i, x_j]$, $i, j = 1, \dots, r$; $i \neq j$ of the original generators under the action of the inner automorphisms defined by the monoms $x_1^{k_1} \dot{x}_2^{k_2} \dots x_r^{k_r}$ (the order of factors does not matter since such automorphism of the commutant, as one can easily check, depends only on the degrees of variables $k_1, k_2, \dots, k_d$ and does not depend on their order). The group Sol_d is isomorphic to the extension of the commutant by the group $\mathbb{Z}^d$ acting on the commutant and some 2-cocycle. (The author follows the terminology in which the extending group is the group which acts by automorphisms rather than the group on which it acts, as is accepted in algebra).

For $d = 2$, the commutant is completely defined, since the described generators of the commutant are not subject to any relations, and *they are the free generators of an abelian group*. But for $d \geq 3$ the description becomes more cumbersome, since the generators are subject to some relations. One can easily see this: for example the element

$$x_1 x_2 x_3 x_1^{-1} x_2^{-1} x_3^{-1}$$

admits two different notations in pairwise commutators of the group generators and adjoined elements, i.e. in the above described generators of the commutant. Easy to see that this element is a cycle on the one-dimensional skeleton of the three-dimensional cube, and there are two ways to decompose it into the product of cycles of plane faces. Thus the use of these generators is not convenient. It was proved in classical works ([9] and others) that this group is embedded into the wreath product $\mathbb{Z}^d \wr \mathbb{Z}^d$, but this embedding is not fit for our purposes either, since the image of these groups under this embedding is not easy to describe.

However, there is a direct way to describe the commutant and the whole group which is partially borrowed from analogies with the *theory of cubical homologies, see f.e. [16].* The same method also allows to describe easily the cocycle defining the extension.

Let us consider the lattice as a one-dimensional complex, i.e. a topological space, namely, as the union of all shifts of the coordinate axes by integer vectors. In another word this is the Caley graph of $\mathbb{Z}^d$ under the ordinary generators as one-dimensional complex. To distinguish it from the lattice as a discrete group, denote this one-dimensional complex by E^d. Consider the additive group $\mathbf{B}_d$ of oriented closed 1-cycles on the space E^d as a one-dimensional complex. We will consider nontrivial in homotopy sense cycles or simply the groups of the first homologies of E^d with integer coefficients - $H_1(E^d) \equiv B_d/Z_d$, where Z_d is the group of the homotopivcally trivial cycles. The generators of $H_1(E^d)$ are *elementary cycles (plackets in the physical terminology, or standard* 1*-cycles in sense of the theory of cubical homologies, see[16])*, i.e. the cycles that go around two-dimensional cell of the lattice. A path around the cell with nodes $(0, e_i, e_i + e_j, e_j)$ in the indicated order, where e_i, e_j are coordinate unit vectors, is called the *elementary* (i, j)*-coordinate cycle.* There is a natural action of the group $\mathbb{Z}^d$ by shifts on the group of homologies $H_1(E^d)$. Each elementary cycle is a translation of one of the elementary coordinate cycles.

Now we define a 2*-cocycle* $\beta(\cdot, \cdot)$ *of the group* $\mathbb{Z}^d$ *with values in the group of homology* $H_1(E^d)$ *or in the group of cycles* $\mathbf{B}_d$, and then the element of the group $H^2(\mathbb{Z}^d; H_1(E^d))$ with respect to action defined above. More exactly, we define at once the cohomology class of cocycles. For this, we associate with every element $v \in \mathbb{Z}^d$ of the lattice an arbitrary connected path τ_v connecting the zero with the element v ("path" as well as "lattice" are understood literally: a path is a continuous mapping of the half-line R_+ or a segment into the lattice as a topological space which sends integer points $\mathbb{N} \in R_+$ to integer vectors, and which is linear at each integer segment). Then, given a pair (v, w) of elements of the lattice $\mathbb{Z}^d$, we define a *cycle* $\beta(v, w) \in \mathbf{B}_d$ formed by three paths:

$$(\tau_v, v + \tau_w, -\tau_{v+w}).$$

This cycle regarded as an element of $\mathbf{B}_d$ is exactly the value of the 2-cocycle $\beta(v, w)$, or more exactly as element of the group $H_1(E^d)$.

Lemma 1 *Different choices of the system of paths* $v \mapsto \tau_v$ *lead to cohomological cocycles.*

Proof. Indeed, if $v \mapsto \tau_v$ and $v \mapsto \rho_v$ are two such systems, then the cycle $(v \mapsto \tau_v, (v \mapsto \rho_v)^{-1})$ realizes the cohomology. ■

As a corollary we obtain that the cocycle correctly defined the class of cohomology, Denote by $\bar{\beta}$ the cohomology class of the constructed cocycle. Note that by construction this cocycle is trivial in the group of paths, however it is non-trivial as a cocycle in the group

of cycles (or homologies) of E^d. So we obtain the extension of infinitely generated abelian group $H_1(E^d)$ by the group $\mathbb{Z}^d$ with the 2-cocycle β. The generators of this group are the genrator of $\mathbb{Z}^d$.

Theorem 1 *The extension of the group $H_1(E^d)$ by the group of shifts $\mathbb{Z}^d$ with the cohomology class $\bar{\beta}$ is canonically isomorphic to the free solvable group of level two with d generators Sol_d. The image of the group $H_1(E^d)$ under this isomorphism is exactly the commutant of the group Sol_d, the image of the elementary (i,j)-coordinate cycle being the commutator of the generators $[x_i, x_j]$, and the action of $\mathbb{Z}^d$ on cycles turning into the action of $\mathbb{Z}^d$ by the inner automorphisms on the commutant of the group Sol_d.*

Proof. Let us define homomorphism from Sol_d to the groups we have defined by putting the generators of Sol_d to the generator of $\mathbb{Z}^d$. It is clear that the commutators if the elements of $\mathbb{Z}^d$ go to the subgorup of cycles (homologies), so we have the surjection of Sol_d onto our group. It is clear also that the kernal is trivial. ∎

Remark 1 *It is possible to prove that $H^2(\mathbb{Z}^r; \mathbf{B}_r) = \mathbb{Z}$, and the constructed cocycle is a generator of this group.*

This construction admits far generalizations. For example, it becomes clear how to understand a continual analogue of the free solvable groups of level two — one should replace the group of cycles on the lattice by an additive group of some 1-cycles on R^d (or De Rham flows), and define the action of R^d and the cocycle exactly as above. Such generalization makes clear the above remark that one should consider the free solvable group as an infinite-dimensional group.

More exactly. Let M a homogeneous space of the Lie group G and $H_1(M)$ is the first homologies with compact support with scalar coefficients (say $\mathbb{C}$). The group G acts on $H_1(M)$, and we can define the cocycle β in the same way: to choose the path γ_x from some fixed point 0 to an arbitrary point x which depends continiously on the point x in the space of pathes, and then define the 2-cocycle with the same formula.

It is interesting to define also the analogue of this construction with smooth differenctial 1-forms instead of H_1. andto define canonical 2-cocycle and element of $H^2(G; \Omega^1(M)$ related to de Rham cohomology. The same technique may be applied to construct the free solvable groups of higher levels — for example, the group of level three may be represented in the same way, since its commutant is the free solvable group with infinite number of generators, hence it is natural to represent the second commutant as the group of 2-cycles on the lattice, and by induction we can constract

$$\mathbb{Z}^d \ltimes_\beta (H_1(E^d) \ltimes_\theta H_1(H_1(E^d)))$$

where $H_1(H_1(E^d))$ is the group of the first homologies on the abelian group $H_1(E^d)$ with respect to *free* generators, and θ the corresponding 2-cocycle of the same type as β above.

3 Space of the pathes and normal forms

We are going to present a general simple technique which reduces the word identity problem in finitely generated groups to the combinatorial geometry on the lattice. We shall see that the word identity problem has a pronounced geometric character. The word identity in a group is equivalent to a notion of equivalence of paths on the lattice which depends on the group; to find a normal form is to solve an isoperimetric problem, etc. In fact, we replace the space of paths on the Cayley graph by a canonically isomorphic space of paths on the lattice. In some cases like the one we study below this method is very efficient.

Given a group G with a system $S = S^{-1}$ of d generators, each its element can be represented by a word in the alphabet S, and with this word we may associate a connected path on the lattice $\mathbb{Z}^{\mathbf{d}}$ starting at zero as follows. Identify the ith generator with the ith coordinate unit vector of a fixed basis of the lattice, $i = 1, \ldots, d$. Now associate with each *oriented* edge of the lattice a generator or its inverse which corresponds to the coordinate axis parallel to this edge, taking the generator, if we pass the edge in the direction of growing distance from zero, and the inverse generator, if the direction is opposite. It is clear, that the space of pathes of the given l (finite or infinite length) in the Cayley graph of the groups with the fixed set of d generators canonically isomorphic to the space of pathes of the same length on the lattices $\mathbb{Z}^d$.

Thus each word is a path on the lattice: the empty word (the unity of the group) turns into the path consisting of one zero point of the lattice, let us call it trivial. Adding some generator (or its inverse) to the end of a given word means adding the oriented edge corresponding to this generator or its inverse, depending on orientation, to the end of the constructed path.

Definition 1 *Two paths are G-equivalent if they define the same element of the group G.*

Thus the word identity problem is reformulated as the problem of G-equivalence of paths on the lattice. Of course this reformulation looks like tauthology, but sometimes it is very useful.

If the relations between generators of the group are generated by the elements of the commutant of the free group, then it suffices to define the equivalence of a closed path (=cycle) to the trivial path (=cycle). This is the case in some of the below examples. But sometimes (e.g. the Heisenberg group) a non-closed path may be equivalent to a closed one. In this case it does not suffice to define the equivalence of closed paths to the trivial one.

Example 1.

Free abelian groups If $G = \mathbb{Z}^{\mathbf{d}}$, then two paths are G-equivalent if their ends coincide.

The group is identified with the lattice. Every closed path (terminating at zero) is equivalent to the trivial one and defines the unity of the group.

Example 2.

Free groups

Another trivial example is given by the free group. In this case two paths are equivalent if they will coincide if we successively cancel in each path all *neighbouring* edges differing only by orientation.

Example 3.

Free nilpotent groups of level 2.

A less trivial example. Let G be the free nilpotent group of level 2 with d generators. First consider the case $d = 2$ — this is the discrete Heisenberg group, i.e. the group of integer-valued upper triangular matrices of third order with ones on the main diagonal. Denote the generators by a, b, then the relations are as follows:

$$[a, b]a = a[a, b], \quad [a, b]b = b[a, b], \quad [a, b] = aba^{-1}b^{-1}.$$

In this case a closed path is equivalent to the zero path if and only if the algebraic (oriented) area enclosed by it is zero. And two paths γ_1 and γ_2 are equivalent if the closed path formed by $\gamma_1\gamma_2^{-1}$ is equivalent to the trivial path.

The same conclusion is also true for the continuous Heisenberg group — the area is exactly the value of the symplectic 2-form (defining the group) on the corresponding 2-cycle. This fact is known for a long time and is used in symplectic geometry and the control theory.

For $d > 2$, the equivalence of a closed path to the zero one is described by the same criterion but applied to all projections of the closed path on the two-dimensional coordinate subspaces of the lattice.

The following key example seems to be new.

Example 4.

Free solvable groups of level two - free metaabelian groups.

Let $G = Sol_d$. By definition, the relations between generators lie in the second (and hence in the first) commutant of the free group. Thus it suffices to define the equivalence of closed cycles to the trivial cycle. But we give the equivalence condition for arbitrary paths.

Given a path γ, an edge ρ of the lattice may occur in γ with different orientation $+, -$. Denote by $\gamma(\rho)$ the algebraic sum (including signs) of all occurrences of this edge in the path γ.

Lemma 2 *Two finite paths γ_1, γ_2 on the lattice $\mathbb{Z}^{\mathbf{d}}$ are Sol_d-equivalent if and only if $\gamma_1(\rho) = \gamma_2(\rho)$ for every edge ρ.*

In other words, the equivalence means that all edges of the lattice occur in both paths with equal multiplicities (taking into account the directions) independently on the order. Recall that in Example 2 (the free group) only neighbouring edges of opposite orientation were canceled.

Proof. The proof follows immediately from the description of the commutant of the group Sol_d. Indeed, since all commutators of Sol_d commute, this means that two paths differing by the order of plackets occurring in them are equivalent. Thus the equivalence class depends only on the total multiplicity of edge taking into account orientations. ∎

This criterion may be easily reformulated as a solution of the word identity problem in the group Sol_d in inner terms of the words, i.e. as a normal form of group elements, but the above criterion is more useful for the sequel. The detale description of this normal form was investigated by student S.Dobrunin. The main preference of this group is the cancellation property for edges - in order to make a label on the given edge we can restrict ourself with the visits of this edge only (and ignore other parts of path). This is not the case for the solvable group of the levels more than two. A nice question -to find the groups with the natural geometrical equvalence of the pathes. The language of the equivalence of the words in initial alphbet could be more cumbersome than the language of the equvalence of the pathes (but of course both are coinsided), The group Sol_d is just an example of that effect. Our interpretation of the classes of the group could be considered as "normal form" of the word - the difference with ordinary normal form is the following: we are givong the *invarianats of the classes - function on the edges - instead of giving some concrete representor of classes.* Of course it is not difficult using our method to give the normal form in usualk sense, but this form won't be minimal.

4 The boundary of the free solvable groups and of the wreath products

Now consider the space of infinite paths on the lattice $\mathbb{Z}^d$ beginning at zero. It may be identified with the space $S^{\mathbb{N}}$ of infinite sequences in the alphabet S of generators and their inverses. We provide it with the natural topology and the Bernoulli measure μ^∞ with equal probability on the generatos and its inverses.

Consider also the space $F(E^d; \mathbb{Z} \cup +\infty) \equiv F^d$ of all functions with integer or infinite values on the set of edges of the lattice E^d, and introduce the mapping Φ of the space of infinite paths $S^{\mathbb{N}}$ in F^d which associates with an infinite path γ and an edge ρ the number $\lim_{n\to\infty} \gamma_n(\rho)$, if the limit exists (i.e. stabilizes), and $+\infty$ otherwise. Here γ_n is the initial segment of the path γ.

We will consider a "simple symmetric" random walk on the free solvable group Sol^d of the level 2 with d generators wich means that the initial measure (transition probability) is a uniform measure with charge $(2d)^{-1}$ on the canonical generators and its inverses. We will describe *the Furstenberg- Poisson boundary* - $\Gamma(Sol^d, \widehat{\mu})$ (see for drfininiton [3] and [1]) of that random walk, using a reduction to the classical simple symmetric random walk on the group $\mathbb{Z}^d$. Namely, each path (trajectory) in the Sol^d can be projected to the $\mathbb{Z}^d$ so, we have a map from space of pathes in Sol^d to the space of pathes $\mathbb{Z}^d$. We will use the map very intensively.

Recall a fundamental fact from the theory of simple random walks on the lattices: if $d = 1, 2$, then the random walk is recurrent, i.e. with probability one it passes infinitely many times through any node and any edge, and if $d > 2$, the random walk is not recurrent, in particular, each edge with probability one occurs in a trajectory of the walk with finite multiplicity (see [7]). Further we shall consider this case $d > 2$. Let us call the functions from F^d stable functions. (We exclude infinite values, since the walk is non-recurrent, thus the values of stable functions are finite.) Two infinite paths with the same image under Φ (finite or not) are called stable equivalent. Thus stable equivalence coincides with the equivalence considered in the previous section. Our aim is to prove that this equivalence coincides with the equivalence defined by the boundary partition σ on $S^{\mathbb{N}}$, see section 3. In other words, we have to prove that there is no measurable function (it suffices to consider functions from L^2) that is orthogonal to all stable functions and is σ-measurable, i.e. is constant on the paths whose finite segments are equivalent in the above sense.

Theorem 2 *1. For $d \le 2$, the space of classes of infinite stable equivalent paths, and the boundary $(\Gamma(Sol_d, \mu), \mu_\sigma)$, where μ is the uniform measure on generators of the group Sol_d, are trivial (*mod0*), i.e. consist of a single point. More exactly, the mapping Φ sends almost all paths to the identically infinite function on edges.*

2. For $d > 2$, the space of classes of stable equivalent paths is not trivial, and it is canonically isomorphic to the boundary $(\Gamma(Sol_d, \mu), \widehat{\mu})$. More exactly, the mapping

$$\Phi : (S^{\mathsf{N}}, \mu^{\infty}) \to (F^d, \widehat{\mu})$$

is well defined, for almost all paths the image is everywhere finite function on edges, and the projection Φ is the canonical isomorphism of this space and the boundary:

$$(\Gamma(Sol_d, \mu), \widehat{\mu}) = (F^d, \widehat{\mu}).$$

here $\widehat{\mu} \equiv \Phi(\mu^{\infty})$

Thus the Poisson–Furstenberg boundary of the pair $\Gamma(Sol_d, \mu)$ is canonically isomorphic as a measure space to the space $(F^d, \widehat{\mu})$ of integer-valued functions on edges provided with the image measure.

Proof. We use a general method which is in more detals published in paper by author [1] in which we had consider the boundary of the groups for the case when so called stable normal form exits. However, we shall make use of the stable equivalence instead of stable normal forms. Again it suffices to prove that the limits of functions depending on stable coordinates exhaust the whole space of σ-measurable (boundary) functions. Assuming that such function exists and is independent on all stable functions, approximate it by cylindric, i.e. n-stable functions depending on stable coordinates of length less than n (see the theorem of section 3). By the same reasons as in general method, such functions must depend only on the coordinates of $(\beta(\gamma))$ that can change when continuing the path γ, and thus, since the walk is non-recurrent, the distance of the corresponding edges β from zero must be greater than some constant depending on n for the paths γ from a set of measure close enough to one also depending on the choice of n. Choosing, as above, a sequence n_k so that these sets of the numbers of coordinates on which depend the approximating functions, do not intersect, we obtain a contradiction (if the functions are not constant) with asymptotic independence of values of the functionals $\beta(\gamma)$ for edges β which are far enough from each other. Thus there is no functions except constants that are orthogonal to all stable functions. Hence the boundary $\Gamma(Sol_d, \mu)$ is mapped to $(F(E^d; \mathbf{Z}), \mathbf{\Phi}(\mu^{\infty}))$ since the image of a path under this mapping depends only on stable functionals of the path, and this mapping is an isomorphism.
■

Remark 2 *The properties of the measure $\Phi(\mu^{\infty})$, i.e. of the canonical measure on the boundary of the free solvable group, are of great interest. This is a measure on the space of integer-valued functions on the edges of the lattice. Its one-dimensional distributions (the values on one edge) can be expressed by the differences of the Green function of the simple*

d-dimensional ($d \geq 3$) random walk at the end and at the beginning of the edge. However, the author knows nothing on the correlations of this natural measure. This measure seems to be more natural than a similar measure on configurations, i.e. on the space of integer-valued functions at the nodes of the lattice which arises in the wreath product model (see below).

Let us apply this method to other groups. Consider the wreath product $\mathbb{Z}^d \wr H \equiv G_d(H)$, where H is a cyclic group of finite or infinite order (now this wreath product is called the "lamplighter group", since the walk on this group is the walk on the lattice with simultaneous random switching lamps in each node of the lattice). This is also a solvable group of level 2 with $d+1$ generators. It is convenient to represent it as the skew product of $\mathbb{Z}^d$ and the space $F_0(\mathbb{Z}^d; H)$ of all finite configurations, i.e. H-valued functions on the lattice. Therefore, this group is naturally represented as a factor group of the group Sol_{d+1}. If $d \geq 3$, then *the boundary $\Gamma(G_d(H), \mu)$, where μ is the uniform measure on generators, is not trivial*, this was first noted in [3]. Easy to see that the boundary can be mapped onto the space $F(\mathbb{Z}^d; H)$ of all configurations: this mapping is just the "final" configuration, i.e. with each node it associates the element of the group H that was "switched on" when the walk visited this node for the last time. However, till now it has not been clear whether this mapping is an isomorphism, i.e. whether each function on the boundary depends only on the final configuration. This question was reduced to some problem on ordinary simple walks on the lattice, but its solution was not obtained. In case when H is a semigroup, the answer is positive ([14]), but the method does not apply to a group.

Corollary 1 *The boundary $\Gamma(G_d(H), \mu)$ is isomorphic to the space $(F(\mathbb{Z}^d; H), \widehat{\mu})$, where $\widehat{\mu}$ is the "final" measure on the space of configurations.*

Proof. One may use the above theorem for the free solvable group Sol_{d+1} and the fact that the wreath product is its factor group. But it is more instructive to carry out a similar argument for the wreath product itself making use of the described method. For simplicity, consider the case $d = 3$ and $H = \mathbb{Z}$, and denote the wreath product $\mathbb{Z}^3 \wr \mathbb{Z}$ by G_3. In this case the wreath product has four generators, the first three of them commute, and the commutators of any element from the subgroup Z^3 formed by these three generators commute with the fourth generator too. Let us identify words in generators with paths on the lattice E^4. It follows from above that two paths are G_3-equivalent if and only if the projections of their ends on the sublattice formed by the first three axes coincide, and the algebraic sum of multiplicities of occurrences of each edge parallel to the fourth axis is the same for both paths. Further argument exactly reproduces the proof of the theorem for the free solvable group. ■

The previous attempts to prove this theorem ran across the following difficulty: reduction of the problem on the boundary of the group G_d to the walk on the group $\mathbb{Z}_d$ (and not

$\mathbb{Z}_{d+1}$) obscures the specific role of the fourth generator. This leads to need to investigate the conditional process and to prove non-triviality of its tail sigma-algebra, and this requires estimations of complicated functionals of trajectories. In the above argument the distinction between generators is seen very well. In particular, *the projection of multiplicities of edges parallel to the fourth axis on the sublattice formed by the first three generators is exactly the final configuration* which was discussed above.

We leave aside other examples, but only note that our method reduces all problems on boundaries of finitely generated groups to problems on sigma-algebras (namely, the sigma-algebras of G-equivalent paths) for the classical random walks on lattices of rank equal to the number of generators of the group. But it is not always easy to describe the G-equivalence of paths (e.g. for the braid groups). This method also shows that to find a normal form is to choose one path from the equivalence class, and the problem of a minimal normal form is *an isoperimetric-like problem: to find a contour of minimal length in a given class of (closed) paths.* For the continuous Heisenberg group this is the classical isoperimetric problem. In the general case these problems were called isoholonomic (see [15]) — these are problems on the minimal length of a curve, given fixed values of some family of 1-forms.

For the free solvable group a normal form of elements also can be described, but it is much more productive to take as coordinates the above considered generators rather than the original ones. It turns out that the description of the boundary does not involve infinite words and even cannot be interpreted in "cylindric" (with respect to $S^{\mathbf{N}}$) terms. We shall consider these questions in details elsewhere.

In conclusion, we note that it is very important to make exact calculation of the main constants — logarithmic volume -

$$v = \lim_{N\mapsto\infty} \frac{\log W_{\leq N}}{N},$$

escape -

$$c = \lim_{N\mapsto\infty} \frac{E_{\mu^{*N}} L(g)}{N},$$

and entropy -

$$h = \lim_{N\mapsto\infty} \frac{H(\mu^{*N})}{N},$$

(where $W_{\leq N}$ is a set of the elements of the groups of the minimal length less or equal to N, μ^{*N} is the N-th convolution of the uniforn measure μ on, $L(g)$ is the minmal length of the element g in those generators, and E_ν – is an expectation with respect to measure ν). Perhaps this calculation for the free solvable groups of the level two – seems to be rather difficult, as well as an explicit calculation of the values of harmonic functions. It is not yet known whether fundamental inequality

$$h \le l \cdot v$$

(see [1]) turns into equality or not.

The techniques we have used to find the boundaries by means of the spaces of paths, and the very realization of groups, have a wide range of applications: the same tools may be used for calculation of other boundaries (f.e.Martin boundary), enumeration of measures with given cocycle, central measures, etc.

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